Linear Algebra I

26/01/2015, Tuesday, 14:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 $(5+4+2+2+2=15 \text{ pts})$ **Linear equations**

Let A and b be given as

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}.$$

Determine

(a) the row echelon form of A.

(b) the reduced row echelon form of A.

(c) the rank of A.

(d) the set of solutions of the homogeneous system $Ax = 0$.

(e) the set of solutions of the system $Ax = b$.

2 $(3+5+7=15 \text{ pts})$ **Matrix multiplication and inverse**

Let A and B be square matrices.

(a) Show that $(A-B)(A+B) = A^2 - B^2$ if and only if $AB = BA$.

(b) Suppose that both A and B are nonsingular. Show that $A^{-1} + B^{-1} = A^{-1}(A+B)B^{-1}$.

(c) Suppose that $A+B$ is nonsingular. Show that $A(A+B)^{-1}B = B(A+B)^{-1}A$.

3 $(3+2+3+3+2+2=15 \text{ pts})$ **From determinants to eigenvalues**

Consider the matrix

$$M = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

(a) Determine the determinant of M.

(b) For which values of a, b, and c, is the matrix M nonsingular?

(c) Determine all values of a, b, and c such that $\text{rank}(M) = 2$.

(d) Determine the characteristic polynomial of M.

(e) Find the eigenvalues of M.

(f) Without computing the eigenvectors, determine whether M is diagonalizable.

4 $(3+3+3+3+3+5 = 20$ pts) **Vector spaces**

Consider the vector space P_4. Let $S \subseteq P_4$ be defined as

$$S := \{p(x) \in P_4 \mid p(x) = x^3 p(\frac{1}{x})\}.$$

(a) Show that S is a subspace of P_4.

(b) Find a basis for S.

(c) Find the dimension of S.

Let $T : P_4 \to P_4$ be defined as

$$T\big(p(x)\big) := p(x) + x^2 p'(\frac{1}{x})$$

where $p'(x)$ denotes the derivative of $p(x)$.

(d) Show that T is a linear transformation.

(e) Determine $\ker(T)$.

(f) Find the matrix representation of T with respect to the ordered basis $\{1, x, x^2, x^3\}$.

5 (10 pts) **Least squares problem**

Find the parabola $y = a + bx + cx^2$ that gives the best least squares approximation to the points:

x	-1	0	1	2
y	1	-1	0	2

6 $(2+4+9 = 15$ pts) **Diagonalization**

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

(a) Show that 1 is an eigenvalue of A.

(b) Determine all eigenvalues of A.

(c) Find a matrix T such that $T^{-1}AT$ is diagonal.

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10 pts free